THE DIFFERENTIAL ANALYTIC INDEX IN SIMONS-SULLIVAN DIFFERENTIAL K-THEORY

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Dedicated to my father Kar-Ming Ho

ABSTRACT. We define the Simons-Sullivan differential analytic index by translating the Freed-Lott differential analytic index via explicit ring isomorphisms between Freed-Lott differential K-theory and Simons-Sullivan differential K-theory. We prove the differential Grothendieck-Riemann-Roch theorem in Simons-Sullivan differential K-theory using a theorem of Bismut.

CONTENTS

1. INTRODUCTION

As explained in [3], [4], [7], [10], the physics motivation for differential K-theory is a quantum field theory whose Largrangian has differential form field strength. This leads to a generalized cohomology theory with a map to ordinary cohomology that implements charge quantization. In [7] Freed argued that there should be a similar extension of topological K-theory. We refer to [8, §1.4] for a historical discussion. The mathematical motivation for differential K-theory can be traced to Cheeger-Simons differential characters [6], the unique differential extension of ordinary cohomology [14], and to the work of Karoubi [11]. It is thus natural to look for differential extensions of generalized cohomology theories, for example topological K-theory. The

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differential extension of topological K-theory is now known as differential K-theory. Roughly speaking, differential K-theory is topological K-theory combined with differential form data in a complicated way, just as differential characters combine ordinary cohomology with differential form data. Various definitions of differential K-theory have been proposed, notably by Bunke-Schick [3], Freed-Lott [8], Hopkins-Singer [10] and Simons-Sullivan [15]. Axioms for differential extensions of generalized cohomology theories are given in [4], where it is shown that two differential extensions of a fixed generalized cohomology theory satisfying certain conditions are uniquely isomorphic. In particular the four models of differential K-theory mentioned above are isomorphic by this abstract result. For more details and an introduction to differential K-theory, see [5].

The Atiyah-Singer family index theorem can be formulated as the equality of the analytic and topological pushforward maps $\mathrm{ind}^{\mathrm{an}} = \mathrm{ind}^{\mathrm{top}} : K(X) \to K(B)$. Applying the Chern character, we get the Grothendieck-Riemann-Roch theorem, the commutativity of the following diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "\mathrm{ind}^{\mathrm{an}}"'] & H^{\mathrm{even}}(X;\mathbb{Q}) \arrow[d, "\int_{X/B}\mathrm{Todd}(X/B)\cup(\cdot)"] \\ K(B) \arrow[r, "\mathrm{ch}"'] & H^{\mathrm{even}}(B;\mathbb{Q}) \end{tikzcd}$$

Analogous theorems hold in differential K-theory. Bunke-Schick prove the differential Grothendieck-Riemann-Roch theorem (dGRR) [3, Theorem 6.19], i.e., for a proper submersion $\pi : X \to B$ of even relative dimension, the following diagram is commutative:

$$\begin{tikzcd} \widehat{K}_{\mathrm{BS}}(X) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{BS}}"] \arrow[d, "\mathrm{ind}^{\mathrm{an}}_{\mathrm{BS}}"'] & \widehat{H}^{\mathrm{even}}(X;\mathbb{R}/\mathbb{Q}) \arrow[d, "\widehat{\int_{X/B}}\widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^VX})*(\cdot)"] \\ \widehat{K}_{\mathrm{BS}}(B) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{BS}}"'] & \widehat{H}^{\mathrm{even}}(B;\mathbb{R}/\mathbb{Q}) \end{tikzcd}$$

where $\widehat{H}(X;\mathbb{R}/\mathbb{Q})$ is the ring of differential characters [6], $\widehat{\mathrm{ch}}_{\mathrm{BS}}$ is the differential Chern character [3, §6.2], $\mathrm{ind}^{\mathrm{an}}_{\mathrm{BS}}$ is the Bunke-Schick differential analytic index [3, §3] and $\widehat{\displaystyle\int_{X/B}}\widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^VX})*$ is a modified pushforward of differential characters [3, §6.4]. The notation is explained more fully in later sections. Freed-Lott prove the differential family index theorem [8, Theorem 7.32] $\mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}} = \mathrm{ind}^{\mathrm{top}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$, where $\mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}}$ and $\mathrm{ind}^{\mathrm{top}}_{\mathrm{FL}}$ are the Freed-Lott differential analytic index [8, Definition 3.11] and the differential topological index [8, Definition 5.33]. Applying the differential Chern character $\widehat{\mathrm{ch}}_{\mathrm{FL}}$ yields the dGRR [8, Corollary 8.23]. Since $\mathrm{ind}^{\mathrm{an}}_{\mathrm{BS}} = \mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}}$ [3, Corollary 5.5], the two dGGR theorems are essentially the same. See [9] for

a short proof of the dGRR.

To this point, the differential index theorem formulated in Simons-Sullivan differential K-theory has not appeared. The purpose of this paper is to fill this gap by both defining the differential analytic index and proving the dGRR in Simons-Sullivan differential K-theory.

The first main result of this paper (Theorem 1) is the construction of explicit ring isomorphisms between Simons-Sullivan differential K-theory and Freed-Lott differential K-theory. While these theories must be isomorphic by [4, Theorem 3.10], the explicit isomorphisms have not been appeared in literature as far as we know. Moreover, it follows from Corollary 1 that the flat parts of these differential K-theories are also isomorphic via the restriction of the explicit ring isomorphisms in Theorem 1. This result is a more explicit version of [4, Theorem 5.5] in this case. The advantage of these explicit ring isomorphisms is that we see which elements in these differential K-groups correspond to each other.

The second main result of this paper is the dGRR in Simons-Sullivan differential K-theory. We first define the Simons-Sullivan differential analytic index by translating the Freed-Lott analytic index via the explicit isomorphisms in Theorem 1. To be precise, we study the special case where the family of kernels $\ker(D^E)$ forms a superbundle. The general case follows from a standard perturbation argument as in [8, §7]. The Simons-Sullivan differential analytic index of an element $\mathcal{E} = [E, h^E, [\nabla^E]] \in \widehat{K}_{\mathrm{SS}}(X)$, in the special case, is given by

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],$$

where $[V, h^V, [\nabla^V]] := \widehat{\mathrm{CS}}^{-1}(\widetilde{\eta}(\mathcal{E}))$, and all the terms will be explained below. The general case of $\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E})$ is given by a similar formula. This formula is considerably more complicated than the Freed-Lott differential analytic index. This indicates that Simons-Sullivan differential K-theory is not the easiest setting for differential index theory, although the Simons-Sullivan construction of the differential K-group is perhaps the simplest among all the existing ones. We then prove the dGRR (Theorem 2) in the special case, i.e., the commutativity of the following diagram

$$\begin{tikzcd} \widehat{K}_{\mathrm{SS}}(X) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{SS}}"] \arrow[d, "\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}"'] & \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q}) \arrow[d, "\widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^V X}) * (\cdot)"] \\ \widehat{K}_{\mathrm{SS}}(B) \arrow[r, "\widehat{\mathrm{ch}}_{\mathrm{SS}}"'] & \widehat{K}_{\mathrm{SS}}(B) \end{tikzcd}$$

in Simons-Sullivan differential K-theory, using a theorem of Bismut [1, Theorem 1.15]. The general case of the dGRR follows by a similar argument,

since [1, Theorem 1.15] can be extended to the general case.

In principle all the theorems and proofs can be transported from Freed-Lott differential K-theory to Simons-Sullivan differential K-theory by the explicit isomorphisms given by Theorem 1. However, with [1, Theorem 1.15] the proof of the dGRR is easier.

The paper is organized as follows: the next two sections contain all the necessary background material. Section 2 reviews Simons-Sullivan differential K-theory. Section 3 reviews Freed-Lott differential K-theory and the construction of the Freed-Lott differential analytic index. The main results of the paper are proved in Section 5, including the explicit isomorphisms between Simons-Sullivan differential K-theory and Freed-Lott differential K-theory, the formula for the differential analytic index in Simons-Sullivan differential K-theory and the dGRR.

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2. Simons-Sullivan differential K-theory

In this section we review Simons-Sullivan differential K-theory [15]. For our purpose, we use the Hermitian version of structured bundles instead of the complex version. Consider a triple (V, h, ∇), where $V \to X$ is a Hermitian vector bundle over a compact manifold X with a Hermitian metric h and a unitary connection ∇. Recall that the Chern character form $\mathrm{ch}(\nabla) \in \Omega^{\mathrm{even}}(X;\mathbb{R})$ and the Chern-Simons transgression form $\mathrm{cs}(\nabla^t) \in \Omega^{\mathrm{odd}}(X;\mathbb{R})$ of two connections ∇^0, ∇^1 on $V \to X$ joined by a smooth curve ∇^t of connections are related by the equality

$$d\,\mathrm{cs}(\nabla^t) = \mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0). \tag{1}$$

Define

$$\mathrm{CS}(\nabla^0, \nabla^1) := \mathrm{cs}(\nabla^t) \mod \mathrm{Im}(d : \Omega^{\mathrm{even}}(X) \to \Omega^{\mathrm{odd}}(X)),$$

where ∇^t is a smooth curve joining the connection ∇^1 and ∇^0. Since $\mathrm{cs}(\nabla^t)$ only depends on the curve joining the connections up to an exact form [15, Proposition 1.6], $\mathrm{CS}(\nabla^0, \nabla^1)$ is well defined [1].

For two connections ∇^0, ∇^1 on $V \to X$, we set $\nabla^0 \sim \nabla^1$ if and only if $\mathrm{CS}(\nabla^0, \nabla^1) = 0$. $\sim$ is an equivalence relation.

The triple $\mathcal{V} = (V, h, [\nabla])$ is called a (Hermitian) structured bundle. Two

[1] It follows from (1) that $d\,\mathrm{CS}(\nabla^0, \nabla^1) = \mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0)$. There are other sign convention, for example see [8]. We will use the convention $d\,\mathrm{CS}(\nabla^0, \nabla^1) = \mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0)$ in this paper.

structured bundles $\mathcal{V} = (V, h^V, [\nabla^V])$ and $\mathcal{W} = (W, h^W, [\nabla^W])$ are isomorphic if there exists a vector bundle isomorphism $\sigma : V \to W$ such that $\sigma^* h^W = h^V$ and $\sigma^*([\nabla^W]) = [\nabla^V]$. Denote by $\mathrm{Struct}(X)$ the set of all isomorphism classes of structured bundles. Direct sum and tensor product of structured bundles are well-defined [15], so $\mathrm{Struct}(X)$ is an abelian semi-ring.

The Simons-Sullivan differential K-group is defined to be

$$\widehat{K}_{\mathrm{SS}}(X) = K(\mathrm{Struct}(X)).$$

Thus, Simons-Sullivan differential K-theory is a K-theory of vector bundles with connections.

To be precise, $[\mathcal{V}_1] - [\mathcal{W}_1] = [\mathcal{V}_2] - [\mathcal{W}_2]$ in $\widehat{K}_{\mathrm{SS}}(X)$ if and only if there exists a structured bundle $(G, h^G, [\nabla^G]) \in \mathrm{Struct}(X)$ such that $V_1 \oplus W_2 \oplus G \cong W_1 \oplus V_2 \oplus G$ and $\mathrm{CS}(\nabla^{V_1} \oplus \nabla^{W_2} \oplus \nabla^G, \nabla^{V_2} \oplus \nabla^{W_1} \oplus \nabla^G) = 0$.

Define

$$\begin{aligned} \mathrm{Struct}_{\mathrm{ST}}(X) &= \{\mathcal{V} \in \mathrm{Struct}(X) | V \text{ is stably trivial}\} \\ \mathrm{Struct}_{\mathrm{SF}}(X) &= \{\mathcal{V} \in \mathrm{Struct}(X) | \mathcal{V} \oplus \mathcal{F} \cong \mathcal{H}\} \end{aligned}$$

where $\mathcal{F} \to X$ and $\mathcal{H} \to X$ are flat structured bundles. Elements in $\mathrm{Struct}_{\mathrm{SF}}(X)$ are said to be stably flat. Let $\mathrm{U} := \varinjlim \mathrm{U}(n)$. Denote by $\theta \in \Omega^1(\mathrm{U}, \mathfrak{u})$ the canonical left invariant $\mathfrak{u}$-valued form on U. Define

$$\begin{aligned} b_j &= \frac{1}{(j-1)!}\left(\frac{1}{2\pi i}\right)^j \int_0^1 (t^2 - t)^{j-1} dt, j \in \mathbb{N} \\ \Theta &= \sum_{j=1} b_j \operatorname{tr}(\overbrace{\theta \wedge \cdots \wedge \theta}^{2j-1}) \in \Omega^{\mathrm{odd}}(\mathrm{U}) \end{aligned}$$

Then define

$$\begin{aligned} \Omega_{\mathrm{U}}(X) &= \{g^*(\Theta) + d\beta | g : X \to \mathrm{U}, \beta \in \Omega^{\mathrm{even}}(X)\} \\ \Omega^{\bullet}_{\mathrm{BU}}(X) &= \{\omega \in \Omega^{\bullet}_{d=0}(X) | [\omega] \in \mathrm{Im}(\mathrm{ch} : K^{-(\bullet \mod 2)}(X) \to H^{\bullet}(X; \mathbb{Q}))\}. \end{aligned}$$

where $\bullet \in \{\mathrm{even}, \mathrm{odd}\}$. The so-called Venice lemma in [15] shows that the map $\widehat{\mathrm{CS}} : \dfrac{\mathrm{Struct}_{\mathrm{ST}}(X)}{\mathrm{Struct}_{\mathrm{SF}}(X)} \to \dfrac{\Omega^{\mathrm{odd}}(X)}{\Omega_{\mathrm{U}}(X)}$ defined by[2]

$$\widehat{\mathrm{CS}}(\mathcal{V}) := \mathrm{CS}(\nabla^V \oplus \nabla^F, \nabla^H) \mod \frac{\Omega_{\mathrm{U}}(X)}{\Omega^{\mathrm{odd}}_{\mathrm{exact}}(X)}$$

is an isomorphism, where $F \to X$ and $H \to X$ are trivial bundles over X such that $H \cong V \oplus F$ and ∇^F, ∇^H are flat connections on F, H, respectively.

[2] This definition differs from the one in [15, Proposition 2.4] by a sign.

Also, the homomorphism

$$\Gamma : \frac{\mathrm{Struct}_{\mathrm{ST}}(X)}{\mathrm{Struct}_{\mathrm{SF}}(X)} \to \widehat{K}_{\mathrm{SS}}(X)$$

defined by $\Gamma(\mathcal{V}) = [\mathcal{V}] - [\dim(\mathcal{V})]$ is injective, for $\dim(\mathcal{V})$ the trivial structured bundle of rank V with the trivial metric and connection. Thus the homomorphism

$$i : \frac{\Omega^{\mathrm{odd}}(X)}{\Omega_{\mathrm{U}}(X)} \to \widehat{K}_{\mathrm{SS}}(X)$$

defined by $i(\phi) = \Gamma \circ \widehat{\mathrm{CS}}^{-1}(\phi)$ is injective. If we pick $\mathcal{V} \in \widehat{\mathrm{CS}}^{-1}(\phi)$, then $\mathcal{V}$ is a stably trivial structured bundle and

$$d\phi = d\,\mathrm{CS}(\nabla^V \oplus \nabla^F, \nabla^H) = \mathrm{ch}(\nabla^V) - \mathrm{rank}(V) \mod \frac{\Omega_{\mathrm{U}}(X)}{\Omega^{\mathrm{odd}}_{\mathrm{exact}}(X)}$$

is independent of the choice of $\mathcal{V}$.

In the following hexagon the diagonal and the off-diagonal sequences are exact, and every square and triangle commutes:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& K^{-1}_{\mathrm{SS}}(X;\mathbb{R}/\mathbb{Z}) \arrow[rr, "B"] \arrow[dr, "j"] & & K(X) \arrow[ur] \arrow[dr, "\mathrm{ch}"] & \\
H^{\mathrm{odd}}(X;\mathbb{R}) \arrow[ur, "i\circ\mathrm{deR}"] \arrow[dr, "\mathrm{deR}"'] & & \widehat{K}_{\mathrm{SS}}(X) \arrow[ur, "\delta"] \arrow[dr, "\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}"] & & H^{\mathrm{even}}(X;\mathbb{R}) \\
& \frac{\Omega^{\mathrm{odd}}(X)}{\Omega_{\mathrm{U}}(X)} \arrow[ur, "i"] \arrow[rr, "d"'] & & \Omega^{\mathrm{even}}_{\mathrm{BU}}(X) \arrow[ur, "\mathrm{deR}"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd} \tag{2}$$

In [15] the homomorphism $\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}} : \widehat{K}_{\mathrm{SS}}(X) \to \Omega^{\mathrm{even}}_{\mathrm{BU}}(X)$ is just denoted by ch, which is a well defined lift of the Chern character form of a connection on a vector bundle to elements in $\widehat{K}_{\mathrm{SS}}(X)$. We use the notation $\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}$ in order to keep track of the Chern character in different usage.

Remark 1. We show that $\Omega_{\mathrm{U}}(X) = \Omega^{\mathrm{odd}}_{\mathrm{BU}}(X)$, and we will use this identification throughout this paper. This is implicitly stated in [13, Diagram 1]. We include the easy proof here for completeness. Let d be the trivial connection on the trivial bundle $X \times \mathbb{C}^N \to X$ for some $N \in \mathbb{N}$. By the proof of [15, Lemma 2.3], the connection $d + g^*(\theta)$ on $X \times \mathbb{C}^N \to X$, where $g : X \to \mathrm{U}$ is an arbitrary but fixed smooth map, has trivial holonomy. Following the proof of [15, Lemma 2.3], we have $g^*(\Theta) = \mathrm{CS}(d, d + g^*(\theta)) = \mathrm{CS}(d, d + g^{-1}dg) =: \mathrm{ch}^{\mathrm{odd}}([g])$, so $\Omega_{\mathrm{U}}(X)\Omega^{\mathrm{odd}}_{\mathrm{BU}}(X)$.

3. Freed-Lott differential K-theory

In this section we review Freed-Lott differential K-theory [8]. If

$$\begin{tikzcd} 0 \arrow[r] & E_1 \arrow[r, "i"] & E_2 \arrow[r, bend left, "j"] & E_3 \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd} \tag{3}$$

is a split short exact sequence of complex vector bundles with connections ∇_i on $E_i \to X$, for $i = 1, 2, 3$, we define the relative Chern-Simons transgression form $\mathrm{CS}(\nabla_1, \nabla_2, \nabla_3) \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$ by

$$\mathrm{CS}(\nabla_1, \nabla_2, \nabla_3) := \mathrm{CS}((i \oplus s)^* \nabla_2, \nabla_1 \oplus \nabla_3),$$

noting that $i \oplus s : E_1 \oplus E_3 \to E_2$ is a vector bundle isomorphism.

The Freed-Lott differential K-group $\widehat{K}_{\mathrm{FL}}(X)$ is defined to be the abelian group with the following generators and relation: a generator of $\widehat{K}_{\mathrm{FL}}(X)$ is a quadruple $\mathcal{E} = (E, h, \nabla, \phi)$, where (E, h, ∇) is as before and $\phi \in \dfrac{\Omega^{\mathrm{odd}}(X)}{\mathrm{Im}(d)}$. The only relation is $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ if and only if there exists a short exact sequence of Hermitian vector bundles (3) and

$$\phi_2 = \phi_1 + \phi_3 - \mathrm{CS}(\nabla_1, \nabla_2, \nabla_3).$$

For $\mathcal{E}_1, \mathcal{E}_2 \in \widehat{K}_{\mathrm{FL}}(X)$, the addition

$$\mathcal{E}_1 + \mathcal{E}_2 := (E_1 \oplus E_2, h^{E_1} \oplus h^{E_2}, \nabla^{E_1} \oplus \nabla^{E_2}, \phi_1 + \phi_2)$$

is well defined. Note that $\mathcal{E}_1 = \mathcal{E}_2$ if and only if there exists $(F, h^F, \nabla^F, \phi^F) \in \widehat{K}_{\mathrm{FL}}(X)$ such that

(1) $E_1 \oplus F \cong E_2 \oplus F$, and
(2) $\phi_1 - \phi_2 = \mathrm{CS}(\nabla^{E_2} \oplus \nabla^F, \nabla^{E_1} \oplus \nabla^F)$,

The Freed-Lott differential Chern character $\widehat{\mathrm{ch}}_{\mathrm{FL}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ is defined by

$$\widehat{\mathrm{ch}}_{\mathrm{FL}}(\mathcal{E}) = \widehat{\mathrm{ch}}(E, \nabla) + i_2(\phi),$$

where $\widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ is the $\mathbb{R}/\mathbb{Q}$ Cheeger-Simons differential characters [6], $\mathcal{E} = (E, h, \nabla, \phi) \in \widehat{K}_{\mathrm{FL}}(X)$, $\widehat{\mathrm{ch}}(E, \nabla)$ is the differential Chern character defined in [6, §4], and $i_2 : \dfrac{\Omega^{\mathrm{odd}}(X)}{\Omega^{\mathrm{odd}}_{\mathbb{Q}}(X)} \to \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ is an injective homomorphism defined by $i_2(\omega)(z) := \displaystyle\int_z \omega \mod \mathbb{Q}$ for $z \in Z_{\mathrm{even}}(X)$ [6, Theorem 1.1].

3.1. **The Freed-Lott differential analytic index.** In this subsection we review the construction of the Freed-Lott differential analytic index. Consider the following diagram:

$$\begin{tikzcd}
(E,h,\nabla) \arrow[ddr] & (S^V X,\widehat{\nabla}^{T^V X}) \arrow[dd, bend left] & (L^V X,\nabla^{L^V X}) \arrow[ddl] & \pi_* E \arrow[dd, bend left] \\
(TX,g^{TX},\nabla^{TX}) \arrow[dr] & (T^V X,g^{T^V X},\nabla^{T^V X}) \arrow[d] & (T^H X,\pi^* g^{TB}) \arrow[dl] & (\ker(D^V),h^{\ker(D^V)},\nabla^{\ker(D^V)}) \arrow[d] \\
 & X \arrow[rr, "\pi"'] & & (B,g^{TB})
\end{tikzcd}$$

In this diagram, $\pi : X \to B$ is a proper submersion with closed fibers of even relative dimension and $T^V X \to X$ is the vertical tangent bundle, which is assumed to have a metric $g^{T^V X}$. $T^H X \to X$ is a horizontal distribution, g^{TB} is a Riemannian metric on B, the metric on $TX \to X$ is defined by $g^{TX} := g^{T^V X} \oplus \pi^* g^{TB}$, ∇^{TX} is the corresponding Levi-Civita connection, and $\nabla^{T^V X} := P \circ \nabla^{TX} \circ P$ is a connection on $T^V X \to X$, where $P : TX \to T^V X$ is the orthogonal projection. $T^V X \to X$ is assumed to have a Spin^c structure. Denote by $S^V X \to X$ the Spin^c-bundle associated to the characteristic Hermitian line bundle $L^V \to X$ with a unitary connection. The connections on $T^V X \to X$ and $L^V X \to X$ induce a connection $\widehat{\nabla}^{T^V X}$ on $S^V X \to X$. Define an even form $\mathrm{Todd}(\widehat{\nabla}^{T^V X}) \in \Omega^{\mathrm{even}}(X)$ by

$$\mathrm{Todd}(\widehat{\nabla}^{T^V X}) = \widehat{A}(\nabla^{T^V X}) \wedge e^{\frac{1}{2} c_1(\nabla^{L^V X})}.$$

The modified pushforward of forms $\pi_* : \Omega^{\mathrm{odd}}(X) \to \Omega^{\mathrm{odd}}(B)$ is defined by

$$\pi_*(\phi) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \phi.$$

The Freed-Lott differential analytic index $\mathrm{ind}^{\mathrm{an}} : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{FL}}(B)$ [8, Definition 3.11] is defined by

$$\mathrm{ind}^{\mathrm{an}}(\mathcal{E}) = (\ker(D^E), h^{\ker(D^E)}, \nabla^{\ker(D^E)}, \pi_*(\phi) + \widetilde{\eta}(\mathcal{E})),$$

where $\mathcal{E} = (E, h, \nabla, \phi) \in \widehat{K}_{\mathrm{FL}}(X)$, $\widetilde{\eta}(\mathcal{E})$ is the Bismut-Cheeger eta form [2] characterized, up to exact form, by

$$d\widetilde{\eta}(\mathcal{E}) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \mathrm{ch}(\nabla) - \mathrm{ch}(\nabla^{\ker(D^E)}),$$

D^E is the family of Dirac operators on $S^V X \otimes E$, and $\ker(D^E)$ is assumed to form a superbundle over B.

4. Main results

4.1. **Explicit isomorphisms between $\widehat{K}_{\mathrm{FL}}$ and $\widehat{K}_{\mathrm{SS}}$.** In this subsection we construct explicit isomorphisms between the Simons-Sullivan differential K-group and the Freed-Lott differential K-group.

Theorem 1. *Let X be a compact manifold. The maps*

$$f : \widehat{K}_{\mathrm{SS}}(X) \to \widehat{K}_{\mathrm{FL}}(X), \qquad g : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{SS}}(X)$$

defined by

$$f([E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]) = (E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0),$$
$$g(E, h^E, \nabla^E, \phi) = [E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],$$

where $\mathcal{V} = (V, h^V, [\nabla^V]) \in \widehat{\mathrm{CS}}^{-1}(\phi)$, are well defined ring isomorphisms, with $f^{-1} = g$. Moreover, f is natural and unique [4, Theorem 3.10], *and is compatible with the structure maps i, j, δ and $\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}$ in (2).*

Proof. First we show that the maps f and g are well defined. For the map f, if $[E_1, h^{E_1}, [\nabla^{E_1}]] - [F_1, h^{F_1}, [\nabla^{F_1}]] = [E_2, h^{E_2}, [\nabla^{E_2}]] - [F_2, h^{F_2}, [\nabla^{F_2}]]$ in $\widehat{K}_{\mathrm{SS}}(X)$, then

$$(E_1, h^{E_1}, \nabla^{E_1}, 0) - (F_1, h^{F_1}, \nabla^{F_1}, 0) = (E_2, h^{E_2}, \nabla^{E_2}, 0) - (F_2, h^{F_2}, \nabla^{F_2}, 0),$$

since there exists $(G, h^G, [\nabla^G]) \in \mathrm{Struct}(X)$ such that $E_1 \oplus F_2 \oplus G \cong F_1 \oplus E_2 \oplus G$ and

$$0 = \mathrm{CS}(\nabla^{E_1} \oplus \nabla^{F_2} \oplus \nabla^G, \nabla^{F_1} \oplus \nabla^{E_2} \oplus \nabla^G) = \mathrm{CS}(\nabla^{E_1} \oplus \nabla^{F_2}, \nabla^{F_1} \oplus \nabla^{E_2}).$$

It follows that the map f is well defined.

For the map g, if $(E, h^E, \nabla^E, \phi) = (F, h^F, \nabla^F, \omega)$ in $\widehat{K}_{\mathrm{FL}}(X)$, then there exists $(G, h^G, \nabla^G, \phi^G) \in \widehat{K}_{\mathrm{FL}}(X)$ such that $E \oplus G \cong F \oplus G$ and $\phi - \omega = \mathrm{CS}(\nabla^F \oplus \nabla^G, \nabla^E \oplus \nabla^G)$. We want

$$\begin{aligned}&[E, h^E, [\nabla^E]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]]\\ &= [F, h^F, [\nabla^F]] + [W, h^W, [\nabla^W]] - [\dim(W), h, [d]],\end{aligned}$$

where $\widehat{\mathrm{CS}}(\mathcal{V}) = \phi$ and $\widehat{\mathrm{CS}}(\mathcal{W}) = \omega$. We need to show that there exists $(G', h^{G'}, [\nabla^{G'}]) \in \mathrm{Struct}(X)$ such that

$$\begin{aligned}&(E, h^E, [\nabla^E]) + (V, h^V, [\nabla^V]) + (\dim(W), h, [d]) + (G', h^{G'}, [\nabla^{G'}])\\ &= (F, h^F, [\nabla^F]) + (W, h^W, [\nabla^W]) + (\dim(V), h, [d]) + (G', h^{G'}, [\nabla^{G'}]),\end{aligned} \tag{4}$$

and $\mathrm{CS}(\nabla^E \oplus \nabla^V \oplus d^W \oplus \nabla^{G'}, \nabla^F \oplus \nabla^W \oplus d^V \oplus \nabla^{G'}) = 0$. (4) is equivalent to

$$\begin{aligned}&(E \oplus V \oplus \dim(W) \oplus G', h^E \oplus h^V \oplus h \oplus h^{G'}, [\nabla^E \oplus \nabla^V \oplus d \oplus \nabla^{G'}])\\ &= (F \oplus W \oplus \dim(V) \oplus G', h^F \oplus h^W \oplus h \oplus h^{G'}, [\nabla^F \oplus \nabla^W \oplus d \oplus \nabla^{G'}]).\end{aligned}$$

Since V and W are stably trivial, there exist trivial bundles V' and W' with connections $\nabla^{V'}$ and $\nabla^{W'}$ such that

$$H^V := \dim(V) \oplus V' = V \oplus V', \qquad H^W := \dim(W) \oplus W' = W \oplus W',$$

and

$$\phi = \mathrm{CS}(\nabla^V \oplus \nabla^{V'}, \nabla^{H^V}), \qquad \omega = \mathrm{CS}(\nabla^W \oplus \nabla^{W'}, \nabla^{H^W}).$$

By taking $G' = G \oplus V' \oplus W'$, we have

$$\begin{aligned} &(E \oplus V \oplus \dim(W)) \oplus (G \oplus V' \oplus W') \\ &\cong (E \oplus G) \oplus (V \oplus V') \oplus (\dim(W) \oplus W') \\ &\cong (F \oplus G) \oplus (\dim(V) \oplus V') \oplus (W \oplus W') \\ &\cong (F \oplus W \oplus \dim(V)) \oplus (G \oplus V' \oplus W') \end{aligned} \tag{5}$$

and for d^V, d^W the trivial connections on $\dim(V)$, $\dim(V)$, respectively,

$$\begin{aligned} &\mathrm{CS}(\nabla^E \oplus \nabla^V \oplus d^W \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^{W'}, \nabla^F \oplus \nabla^W \oplus d^V \oplus \nabla^G \oplus \nabla^{V'} \oplus \nabla^{W'}) \\ &= \mathrm{CS}(\nabla^E \oplus \nabla^G, \nabla^F \oplus \nabla^G) + \mathrm{CS}(\nabla^V \oplus \nabla^{V'}, d^V \oplus \nabla^{V'}) + \mathrm{CS}(d^W \oplus \nabla^{W'}, \nabla^W \oplus \nabla^{W'}) \\ &= -\phi + \omega + \mathrm{CS}(\nabla^V \oplus \nabla^{V'}, \nabla^{H^V}) + \mathrm{CS}(\nabla^{H^W}, \nabla^W \oplus \nabla^{W'}) \\ &= -\phi + \omega + \phi - \omega \\ &= 0 \end{aligned} \tag{6}$$

(5) and (6) imply (4), so the map $g : \widehat{K}_{\mathrm{FL}}(X) \to \widehat{K}_{\mathrm{SS}}(X)$ is well defined.

We now show that f and g are inverses. Note that

$$\begin{aligned} (g \circ f)([E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]]) &= g((E, h^E, \nabla^E, 0) - (F, h^F, \nabla^F, 0)) \\ &= [E, h^E, [\nabla^E]] - [F, h^F, [\nabla^F]] \end{aligned}$$

as $\widehat{\mathrm{CS}}^{-1}(0) = 0 \in \dfrac{\mathrm{Struct}_{\mathrm{ST}}(X)}{\mathrm{Struct}_{\mathrm{SF}}(X)}$. For the other direction, we consider

$$(f \circ g)(E, h^E, \nabla^E, \phi) = (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0) - (\dim(V), h, d, 0),$$

where $\widehat{\mathrm{CS}}(\mathcal{V}) = \phi$ for $\mathcal{V} := (V, h^V, [\nabla^V]) \in \mathrm{Struct}_{\mathrm{ST}}(X)$. It suffices to show

$$(E, h^E, \nabla^E, \phi) + (\dim(V), h, d^V, 0) = (E, h^E, \nabla^E, 0) + (V, h^V, \nabla^V, 0),$$

which is equivalent to

$$(E \oplus \dim(V), h^E \oplus h, \nabla^E \oplus d^V, \phi) = (E \oplus V, h^E \oplus h^V, \nabla^E \oplus \nabla^V, 0). \tag{7}$$

To see this, since $\mathcal{V} = (V, h^V, [\nabla^V])$ is stably trivial, there exist trivial structured bundles $\mathcal{F} = (F, h, [d^F])$ and $\mathcal{H} = (H, h, [d^H])$ such that $V \oplus F \cong H$ and $\phi = \mathrm{CS}(d^H, \nabla^V \oplus d^F)$. Thus $E \oplus \dim(V) \oplus \dim(F) \cong E \oplus \dim(H) \cong E \oplus V \oplus F$, and

$$\begin{aligned} \mathrm{CS}(\nabla^E \oplus \nabla^V, \nabla^E \oplus d^V) &= \mathrm{CS}(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^V \oplus d^F) \\ &= \mathrm{CS}(\nabla^E \oplus \nabla^V \oplus d^F, \nabla^E \oplus d^H) = \phi. \end{aligned}$$

This proves (7).

f is obviously a natural ring homomorphism. Since $g = f^{-1}$, g is also a ring homomorphism. $\square$

The following corollary follows from the compatibility of f and $\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}$.

Corollary 1. *Let X be a compact manifold. The following diagram is commutative.*

$$\begin{tikzcd}
0 \arrow[r] & K^{-1}_{\mathrm{SS}}(X;\mathbb{R}/\mathbb{Z}) \arrow[r, "j"] \arrow[d, "\bar{f}"'] & \widehat{K}_{\mathrm{SS}}(X) \arrow[r, "\mathrm{ch}"] \arrow[d, "f"'] & \Omega_{\mathrm{BU}}(X) \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & K^{-1}_{\mathrm{L}}(X;\mathbb{R}/\mathbb{Z}) \arrow[r, "j'"'] & \widehat{K}_{\mathrm{FL}}(X) \arrow[r, "\omega"'] & \Omega_{\mathrm{BU}}(X) \arrow[r] & 0
\end{tikzcd}$$

where $\bar{f}$ is the restriction of f to $K^{-1}_{\mathrm{SS}}(X;\mathbb{R}/\mathbb{Z})$. Here $\omega : \widehat{K}_{\mathrm{FL}}(X) \to \Omega_{\mathrm{BU}}(X)$ is defined by $\omega(E, h^E, \nabla^E, \phi) = \mathrm{ch}(\nabla^E) + d\phi$.

Note that the horizontal sequences are exact by [8], [15].

4.2. **The differential analytic index in $\widehat{K}_{\mathrm{SS}}$.** In this subsection we give the formula for the differential analytic index in Simons-Sullivan differential K-theory.

Let $\pi : X \to B$ be a proper submersion of even relative dimension and its fibers are assumed to be Spin^c. The differential analytic index in Simons-Sullivan differential K-theory is given by forcing the following diagram to be commutative:

$$\begin{tikzcd}
\widehat{K}_{\mathrm{SS}}(X) \arrow[r, "f"] \arrow[d, "\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}"'] & \widehat{K}_{\mathrm{FL}}(X) \arrow[d, "\mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}}"] \\
\widehat{K}_{\mathrm{SS}}(B) & \widehat{K}_{\mathrm{FL}}(B) \arrow[l, "g"]
\end{tikzcd}$$

Let $\mathcal{E} := [E, h^E, [\nabla]] \in \widehat{K}_{\mathrm{SS}}(X)$. Since

$$\begin{aligned}(g \circ \mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}} \circ f)(\mathcal{E}) = {}& [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] \\ &+ [V, h^V, [\nabla^V]] - [\dim(V), h, [d]],\end{aligned}$$

where $\mathcal{V} := (V, h^V, [\nabla^V]) \in \dfrac{\mathrm{Struct}_{\mathrm{ST}}(B)}{\mathrm{Struct}_{\mathrm{SF}}(B)}$ is uniquely determined by the condition $\widehat{\mathrm{CS}}(\mathcal{V}) = \widetilde{\eta}(\mathcal{E}) \mod \dfrac{\Omega_{\mathrm{U}}(B)}{\Omega^{\mathrm{odd}}_{\mathrm{exact}}(B)}$, it follows that the differential analytic index in the Simons-Sullivan differential K-theory $\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}} : \widehat{K}_{\mathrm{SS}}(X) \to \widehat{K}_{\mathrm{SS}}(B)$ is given by

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]], \tag{8}$$

where $\ker(D^E)$ is assumed to form a superbundle over B. Although $\mathcal{V} := \widehat{\mathrm{CS}}^{-1}(\widetilde{\eta}(\mathcal{E}))$ is uniquely determined up to a stably flat structured bundle, its class $[\mathcal{V}] \in \widehat{K}_{\mathrm{SS}}(B)$ is unique since the differential K-theory class of a stably flat structured bundle is zero. Moreover, since $\mathrm{ind}^{\mathrm{an}}_{\mathrm{FL}}$ is well defined (see [9, Proposition 1] for a proof which does not use the differential family index theorem), it follows that $\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}$ is well defined too.

If one defines the Simons-Sullivan differential analytic index $\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}$ without considering the other differential analytic indices, a natural candidate would be, say, in the special case when $\ker(D^E) \to B$ is a superbundle,

$$\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E}) = [\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]].$$

This definition coincides with (8) if and only if $\mathcal{V} \in \mathrm{Struct}_{\mathrm{SF}}(B)$, which is equivalent to saying that $\widetilde{\eta}(\mathcal{E}) \in \Omega_{\mathrm{U}}(B) = \Omega^{\mathrm{odd}}_{\mathrm{BU}}(B)$. However, this is not true since

$$d\widetilde{\eta}(\mathcal{E}) = \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \mathrm{ch}(\nabla) - \mathrm{ch}(\nabla^{\ker(D^E)}),$$

which shows that $\widetilde{\eta}(\mathcal{E})$ is not closed in general.

Lemma 1. *Let* $\mathcal{E} = [E, h, [\nabla]] \in \widehat{K}_{\mathrm{SS}}(X)$. *Then*

$$\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E})) = \mathrm{ch}(\nabla^{\ker(D^E)}) + d\widetilde{\eta}(\mathcal{E}).$$

It follows from Lemma 1 and the local family index theorem that

$$\begin{aligned}\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E})) &= \mathrm{ch}(\nabla^{\ker(D^E)}) + d\widetilde{\eta}(\mathcal{E}) \\ &= \int_{X/B} \mathrm{Todd}(\widehat{\nabla}^{T^V X}) \wedge \mathrm{ch}(\nabla^E) \\ &= \pi_*(\mathrm{ch}_{\widehat{K}_{\mathrm{SS}}}(\mathcal{E})).\end{aligned}$$

We define the Simons-Sullivan differential Chern character $\widehat{\mathrm{ch}}_{\mathrm{SS}} : \widehat{K}_{\mathrm{SS}}(X) \to \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ by

$$\widehat{\mathrm{ch}}_{\mathrm{SS}}(\mathcal{E}) := \widehat{\mathrm{ch}}(E, \nabla),$$

where $\mathcal{E} = [E, h, [\nabla]]$.

It is instructive to note that the following diagram commute,

$$\begin{tikzcd} & \widehat{K}_{\mathrm{SS}}(X) \arrow[dd, "f"] \\ \dfrac{\Omega^{\mathrm{odd}}(X)}{\Omega^{\mathrm{odd}}_{\mathrm{BU}}(X)} \arrow[ur, "i"] \arrow[dr, "j"'] & \\ & \widehat{K}_{\mathrm{FL}}(X) \end{tikzcd}$$

where $f : \widehat{K}_{\mathrm{SS}}(X) \to \widehat{K}_{\mathrm{FL}}(X)$ is the isomorphism given by Theorem 1.

4.3. **The differential Grothendieck-Riemann-Roch theorem.** In this subsection we prove the dGRR in Simons-Sullivan differential K-theory. To be precise, we first prove the special case that the family of kernels $\ker(D^E)$ forms a superbundle by a theorem of Bismut reviewed below. The general case follows from the standard perturbation argument as in [8, §7].

We now recall Bismut's theorem. For the geometric construction of the analytic index given in §4.2, with the fibers assumed to be Spin, and $\ker(D^E) \to B$ assumed to form a superbundle, we have

$$\widehat{\mathrm{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\widetilde{\eta}) = \widehat{\int_{X/B}} \widehat{\widehat{A}}(T^V X, \nabla^{T^V X}) * \widehat{\mathrm{ch}}(E, \nabla^E) \tag{9}$$

[1, Theorem 1.15], where $\widehat{\int_{X/B}}$ is the pushforward of differential characters for proper submersion [8, §8.3], $*$ is the multiplication of differential characters [6, §1], and $\widehat{\widehat{A}}(T^V Z, \nabla^{T^V X}) \in \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ is the differential character associated to the $\widehat{A}$-class (see [6, §2]). If the fibers are assumed to be Spin^c, (9) has the obvious modification, and in our notation becomes

$$\widehat{\mathrm{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\widetilde{\eta}) = \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) * \widehat{\mathrm{ch}}(E, \nabla^E), \tag{10}$$

for $\widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X}) \in \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q})$ the differential character associated to the Todd class (see [6, §2]). We will write $\widehat{\mathrm{Todd}}(T^V X, \widehat{\nabla}^{T^V X})$ as $\widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^V X})$ in the sequel. Note that (9) and (10) extend to the general case where $\ker(D^E) \to B$ does not form a bundle [1, p. 23].

Theorem 2 (**Differential Grothendieck-Riemann-Roch theorem).** *Let $\pi : X \to B$ be a proper submersion with closed* Spin^c*-fibers of even dimension. The following diagram is commutative:*

$$\begin{array}{ccc}
\widehat{K}_{\mathrm{SS}}(X) & \xrightarrow{\widehat{\mathrm{ch}}_{\mathrm{SS}}} & \widehat{H}^{\mathrm{even}}(X; \mathbb{R}/\mathbb{Q}) \\
{\scriptstyle \mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}}\Big\downarrow & & \Big\downarrow {\scriptstyle \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^V X}) * (\cdot)} \\
\widehat{K}_{\mathrm{SS}}(B) & \xrightarrow[\widehat{\mathrm{ch}}_{\mathrm{SS}}]{} & \widehat{H}^{\mathrm{even}}(B; \mathbb{R}/\mathbb{Q})
\end{array}$$

i.e., if $\mathcal{E} = [E, h, [\nabla^E]] \in \widehat{K}_{\mathrm{SS}}(X)$, then

$$\widehat{\mathrm{ch}}_{\mathrm{SS}}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E})) = \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(\nabla^{T^V X}) * \widehat{\mathrm{ch}}_{\mathrm{SS}}(\mathcal{E}).$$

Proof.

$$\begin{aligned}
&\widehat{\mathrm{ch}}_{\mathrm{SS}}(\mathrm{ind}^{\mathrm{an}}_{\mathrm{SS}}(\mathcal{E}))\\
&= \widehat{\mathrm{ch}}_{\mathrm{SS}}([\ker(D^E), h^{\ker(D^E)}, [\nabla^{\ker(D^E)}]] + [V, h^V, [\nabla^V]] - [\dim(V), h, [d]])\\
&= \widehat{\mathrm{ch}}(\ker(D^E), \nabla^{\ker(D^E)}) + i_2(\widetilde{\eta}(\mathcal{E}))\\
&= \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^V X}) * \widehat{\mathrm{ch}}(E, \nabla^E)\\
&= \widehat{\int_{X/B}} \widehat{\mathrm{Todd}}(\widehat{\nabla}^{T^V X}) * \widehat{\mathrm{ch}}_{\mathrm{SS}}(\mathcal{E})
\end{aligned}$$

where the second equality follows from Proposition **??** and the third equality follows from (10). □

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